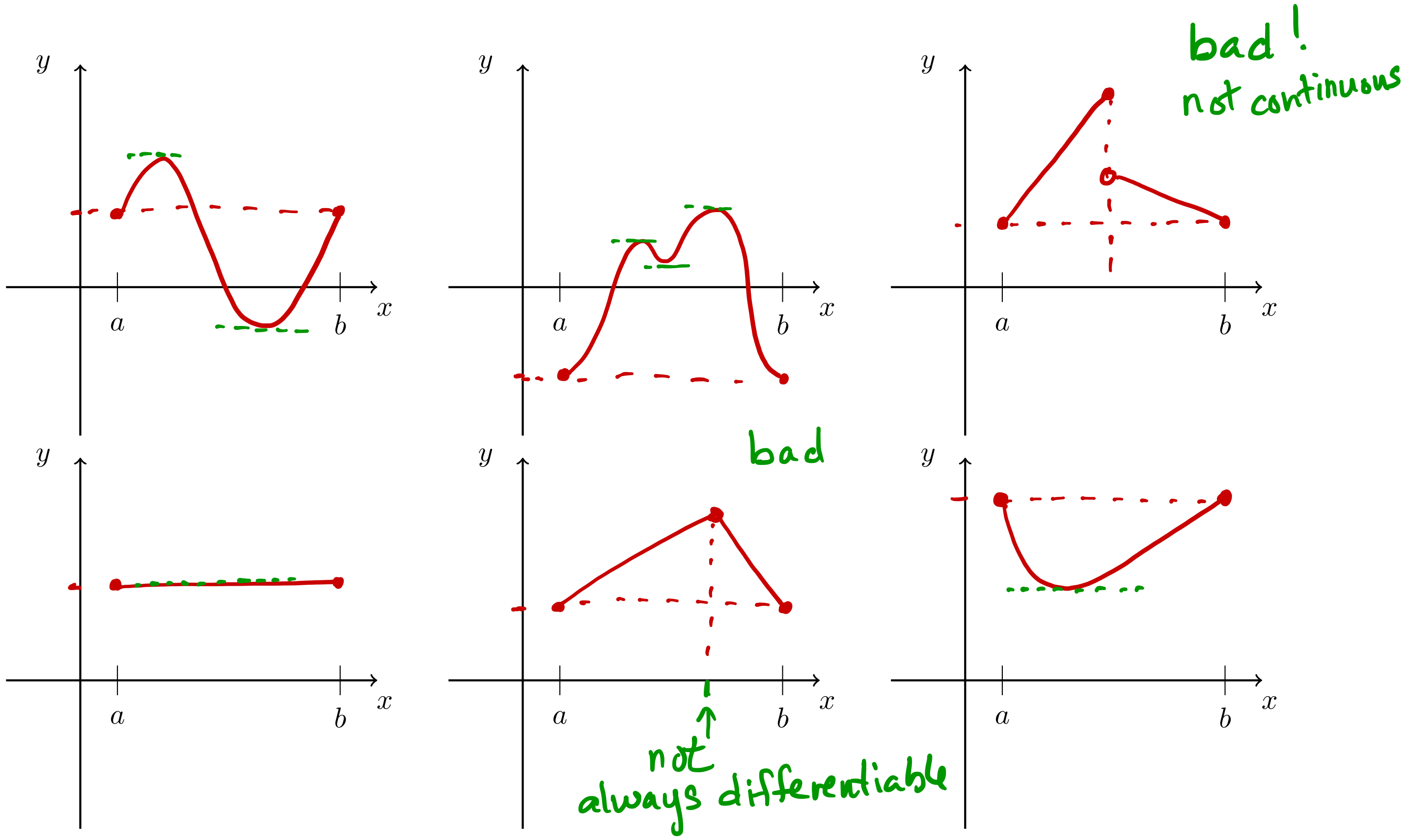
LECTURE NOTES: 4-2 THE MEAN VALUE THEOREM (PART 1)

MOTIVATING EXAMPLES: Draw several examples of graphs of functions such that (i) the domain is $[a, b]$ and (ii) $f(a) = f(b)$. Note you are not *required* to make sketches that are continuous or differentiable, though you may choose to do so.

Question 4 in Green



QUESTION 1: What does it mean to call something a *Theorem* in a mathematics course?

It's a statement that is always true provided all hypotheses (ie the "IF" part) hold and it's possible to prove the statement always holds.

i.e. – a pattern that seems to hold

– an argument that shows it always holds.

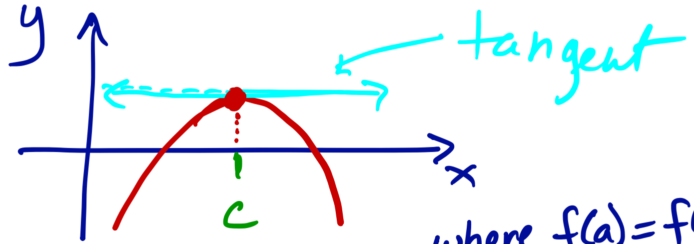
QUESTION 2: What is the difference between a *conjecture* and a *Theorem* in a mathematics course?

conjecture: A pattern (or rule) you think holds.

theorem: A pattern (or rule) you think holds along with an explanation that proves you're right.

QUESTION 3: State in plain old English (or draw a picture) to explain what it means for the graph of $f(x)$ if you know $f'(c) = 0$.

The tangent to curve is horizontal at $x = c$.



where $f(a)=f(b)$

QUESTION 4: Based on our examples on the previous page and your knowledge of graphs more broadly, what requirements would be needed to *guarantee* the existence of an x-value c in the open interval (a, b) such that $f'(c) = 0$?

You're gonna have a problem if f isn't continuous or has points of nondifferentiability.

nine syllables! :)

ROLLE'S THEOREM: If

- $f(x)$ is continuous on $[a,b]$

  and
- $f(x)$ is differentiable on (a,b)

  and
- $f(a)=f(b)$,

then there is a number c in the interval (a, b) such that $f'(c) = 0$.

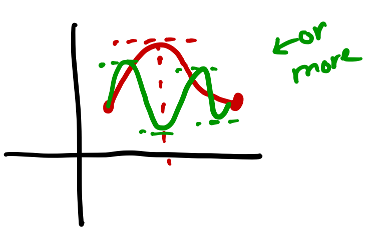
QUESTION 5: Now that we see a pattern, can we give an argument for why that pattern should hold? (HINT: What does the Extreme Value Theorem say again??)

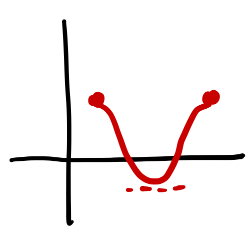
argument

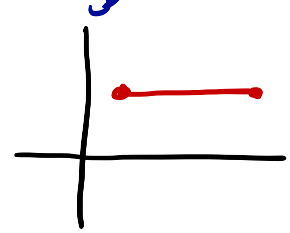
Since $f(x)$ is continuous, EVThm implies $f(x)$ has an absolute maximum and absolute minimum.

Since $f(x)$ is differentiable, these extreme values would have to occur at "turn around" points (where $f'(x)=0$) or end points.

But the end points have the same y-value! So ... there's got to be a max elsewhere, a min elsewhere, or... (tricky) the max = min.







PRACTICE PROBLEMS:

1. Consider $f(x) = x^3 - 2x^2 - 4x + 2$ on the interval $[-2, 2]$.

   (a) Verify that the function $f(x)$ satisfies the hypothesis of Rolle's Theorem on the given interval.

Since $f(x)$ is a polynomial, it's continuous and differentiable everywhere. So it's certainly continuous on $[-2,2]$ and differentiable on $(-2,2)$.

Note: Complete sentences in English w/ proper punction.

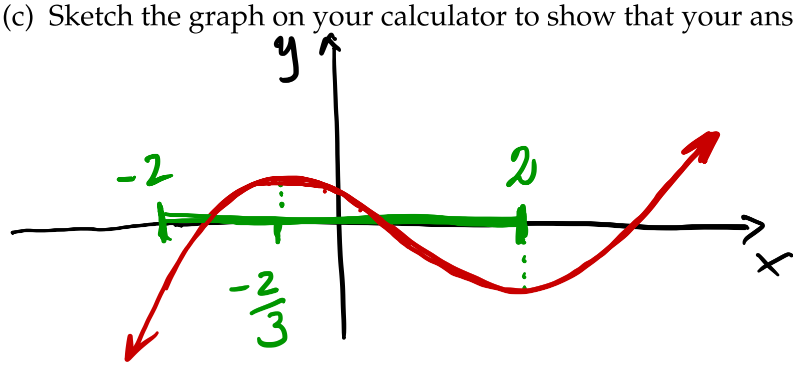
   (b) Find all numbers c that satisfy the conclusion of Rolle's Theorem.

thinking:
Need x-values where,
① $f' = 0$
and
② x in $(-2,2)$

work:
$f'(x) = 3x^2 - 4x - 4 = (3x+2)(x-2)$
$f'(x) = 0$ when $x = \frac{-2}{3}$ or $x = 2$ ← Not in $(-2,2)$

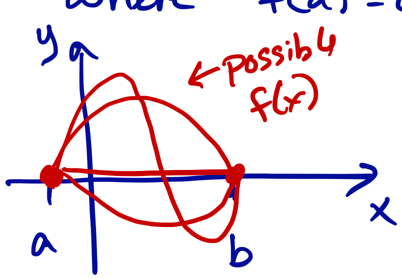
answer: $c = -2/3$

   (c) Sketch the graph on your calculator to show that your answer above are correct.



2. *Use Rolle's Theorem* to show that the equation $x^3 - 15x + d = 0$ can have at most one solution in the interval $[-2, 2]$.
   HINT: Show that there is no way there could be two solutions!

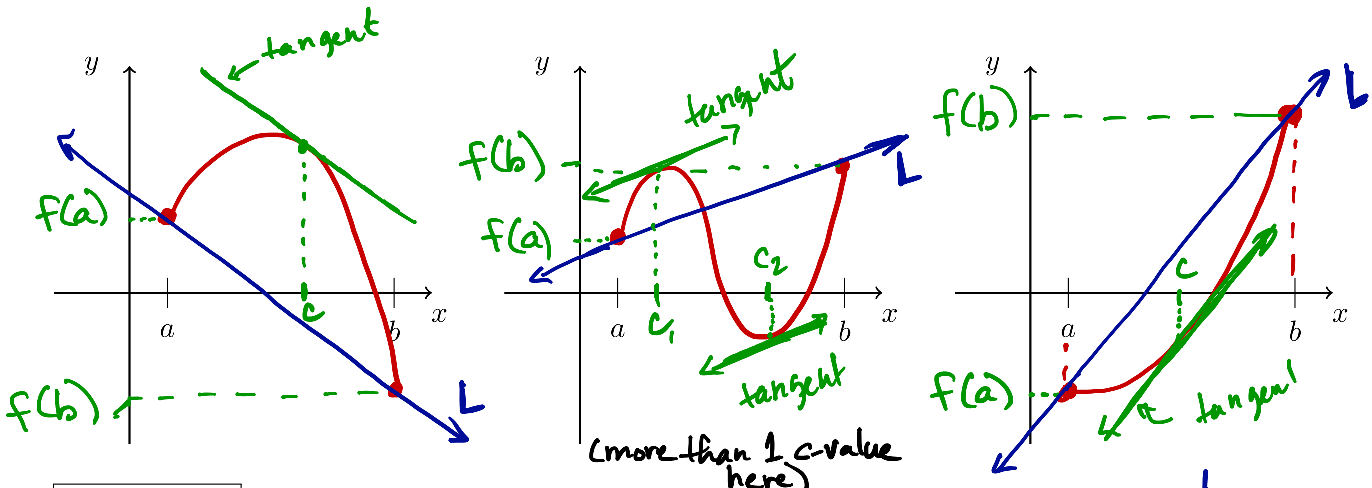
OK. I'll follow the hint. What if $f(x) = x^3 - 15x + d$ has two solutions in $[-2,2]$? Then $f(x)$ would have two x-values (say $x=a$ + $x=b$) where $f(a) = 0 = f(b)$. [Rolle's Thm uses $\boxed{f'}$ so I'll find that.]



Now $f'(x) = 3x^2 + 15 = 0$ if $x = \pm\sqrt{5}$. But neither $\sqrt{5}$ or $-\sqrt{5}$ are in $[-2,2]$. So $f(x)$ has no turn around points. So it can't have two solutions.

MOTIVATING EXAMPLES: Draw several examples of graphs of functions such that (i) the domain is $[a, b]$, (ii) $f(x)$ is continuous on $[a, b]$, **and** (iii) $f(x)$ is differentiable on $[a, b]$. We are *not* assuming that $f(a) = f(b)$.

no jumps smooth



QUESTION 6: In each picture above, draw (or in some other way identify) the quantity: L

$$\frac{f(b) - f(a)}{b - a}. = m = \text{slope of line between the points } (a, f(a)) \text{ and } (b, f(b)).$$

What would this quantity be if Rolle's Theorem applied?

If $f(a) = f(b)$, this line is horizontal.
So $m = 0$.

THE MEAN VALUE THEOREM: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(intuitively) there is some x-value c in (a,b) where slope of tangent (in green) is the same as slope of line between endpoints (the line L in blue)

OBSERVATION: The Mean Value Theorem is just Rolle's Theorem if you turn your head sideways.

alternately: picture the x-axis as parallel to line L (in blue).

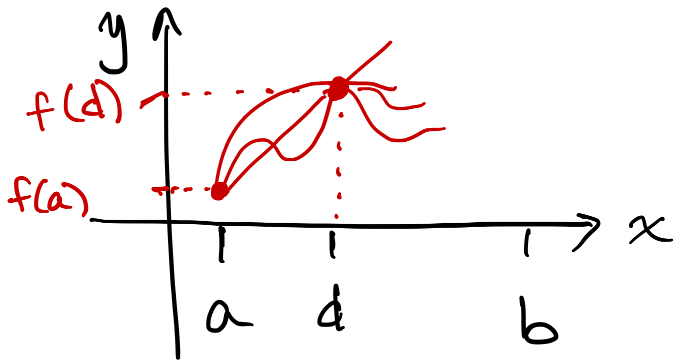
rigorously: Make a new function

$$h(x) = f(x) - L(x),$$

then apply Rolle's Thm.

ooh! Does Rolle's Thm apply?

QUESTION 7: Assume that $f(x)$ is continuous and differentiable on the interval $[a, b]$ and assume there exists some x-value d in (a, b) such that $f(d) > f(a)$, can you draw any conclusion about $f'(x)$? Why or why not?



There's got to be some places where $f'(x)$ is positive. (Alternately, apply MVThm to $[a,d]$)

THEOREM 5: If $f'(x) = 0$ for all x in the interval (a, b), then

$f(x) = c$ on (a,b).

alternately → the graph of $f(x)$ is horizontal or constant or flat on (a,b).

QUESTION 8: How would you explain why this theorem is true? (Hint: See your answer to Question 7!)

The answer to Question 7 shows that if $f(x)$ was not constant, then $f'(x)$ would not always be zero. That is, $f(d) > f(a)$ forces $f'(x) > 0$ somewhere. Similarly, $f(d) < f(a)$ forces $f'(x) < 0$ somewhere.

QUESTION 9: If $f(x)$ gives the *position* of an object as a function of time, what "common sense" idea is the MVT telling us? Theorem 5?

If $f(x)$ gives position over time then

$f'(x)$ is instantaneous velocity and

$\dfrac{f(b)-f(a)}{b-a}$ is average velocity from time a to time b.

So $\boxed{\text{MVThm}}$ says there must be some time $\underline{\underline{c}}$ where a body's instantaneous velocity equals its average velocity.

So $\boxed{\text{Thm 5}}$ says if a body's velocity is zero over some time interval, it's position is constant (i.e. it ain't moving!)